Term structures of implied volatilities: Absence of arbitrage and existence results

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Motivation: market models of implied volatilities

- European vanilla options are traded liquidly on the market
- Market prices of these options are not consistent with Black-Scholes model
- Alternative stock price models...
  - Stochastic volatility models
  - Lévy models
  - etc...

... can reproduce market prices of European vanillas, but:

  - Calibration to market prices is often not easy
  - Market incompleteness

- **Idea:** Specify a joint model for stock and option prices.
  - Denote by $\hat{\sigma}(T, K)$ the implied volatility of a European call $C(K, T)$ with strike K and maturity T
  - Consider a model with bank account, stock S and a set of call options $C(K, T)$ on S as underlying assets. For an m-dimensional P-Brownian motion W, we assume dynamics

$$\frac{dS_t}{S_t} = \mu_t dt + \sigma_t dW_t$$
$$d\hat{\sigma}_t(T, K) = u_t(T, K)dt + v_t(T, K)dW_t$$

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- **Aims:**
    - Find necessary and sufficient conditions on the coefficients μ_t, σ_t, $u_t(T, K)$, $v_t(T, K)$ for absence of arbitrage in this model.
    - Find classes of examples for such coefficients.

Outline of the talk

1. Introduction
2. No-arbitrage conditions in implied volatility models
    1. Implied volatilities for convex payoff functions
    2. Drift restrictions
    3. Example: Call option
3. On strong solutions of SDEs
4. A class of arbitrage-free term structure models for implied volatilities

No-arbitrage conditions in implied volatility models

Implied volatilities for convex payoff functions

Let h be a (sufficiently integrable) convex payoff function. In the Black-Scholes model, the price C_t^T of an option paying $h(S_T)$ at time T is given by $c(S_t,(T-t)\sigma^2)$, where $c(S,\Upsilon)$ denotes the solution of

$$\left.\begin{array}{rlr} \frac{1}{2}S^2 c_{SS}(S,\Upsilon) &= c_\Upsilon(S,\Upsilon) & (S,\Upsilon>0) \\ c(S,0) &= h(S) & (S>0) \end{array}\right\}$$

Definition (Implied volatility and forward implied volatility)

For a given arbitrage-free family of option prices C_t^T for all $T > 0$, we can define the *implied volatility* to be the unique parameter $\hat{\sigma}(t, T) \geq 0$ satisfying

$$c\Big(S_t, (T-t)\hat{\sigma}^2(t,T)\Big) = C_t^T.$$

If the function $T \mapsto C_t^T$ is differentiable in T, we define the *forward implied volatility* for the maturity T by

$$X(t,T) := \frac{\partial}{\partial T}\left((T-t)\hat{\sigma}^2(t,T)\right).$$

Under the no-arbitrage assumption, we have $X(t,T) \geq 0$.

Drift restrictions

Let c be the BS price function, W m-dim BM, $\mathbb{F}$ generated by W. We model a stock price process $(S_t)_{t\geq 0}$ and a family of price processes $(C_t^T)_{0\leq t\leq T}$ $(T>0)$ of contracts paying $h(S_T)$ at time T by

$$C_t^T = c\left(S_t, \int_t^T X(t,s)ds\right)$$

with dynamics

$$\begin{aligned} dS_t &= \mu_t S_t dt + \sigma_t S_t dW_t^1 && (t\geq 0), && S_0 = s_0, \\ dX(t,T) &= \alpha(t,T)dt + v(t,T)dW_t && (0\leq t\leq T), && X(0,T) = X_0(T) \end{aligned}$$

Here $X(\cdot,T)$ is a nonnegative process modelling forward implied volatility for maturity T.

Theorem (No arbitrage conditions)

There exists a common loc. mart. meas. $Q \approx P$ *for* $(S_t)_{t\geq 0}$ *and all* $(C_t^T)_{0\leq t\leq T}$, $T>0$, *iff for some (int.)* $\mathbb{R}^m$*-valued process* b

$$\begin{aligned}\sigma_t^2 + \sigma_t \lim_{T\searrow t}\left(\frac{S_t c_{S\Upsilon}}{c_\Upsilon}\int_t^T v^1(t,s)ds\right) &+ \\ + \frac{1}{2}\lim_{T\searrow t}\left(\frac{c_{\Upsilon\Upsilon}}{c_\Upsilon}\left|\int_t^T v(t,s)ds\right|^2\right) &- X(t,t) = 0,\end{aligned} \tag{1}$$

$$\mu_t = -\sigma_t b_t^1, \tag{2}$$

$$\begin{aligned}&\alpha(t,T) = -b_t\cdot v(t,T)\\ &-\frac{c_{\Upsilon\Upsilon}}{c_\Upsilon}v(t,T)\cdot\int_t^T v(t,s)ds - \frac{1}{2}\partial_\Upsilon\left(\frac{c_{\Upsilon\Upsilon}}{c_\Upsilon}\right)X(t,T)\left|\int_t^T v(t,s)ds\right|^2\\ &-S_t\frac{c_{S\Upsilon}}{c_\Upsilon}\sigma_t v^1(t,T) - S_t\partial_\Upsilon\left(\frac{c_{S\Upsilon}}{c_\Upsilon}\right)X(t,T)\sigma_t\int_t^T v^1(t,s)ds.\end{aligned} \tag{3}$$

Remarks.

- The equations (1) - (3) are the analogous to the HJM drift restrictions in interest modelling.
- The free input parameters of the model are the processes $v(\cdot, T)$ and b. They determine σ, μ and $\alpha(\cdot, T)$ via (1) - (3).
- If we take $v(\cdot, T) = 0 \; \forall T$, then also $\alpha(\cdot, T) = 0 \; \forall T$. Hence $X(t, T) = X_0(T)$ and $\sigma_t^2 = X_0(t)$. This is the Black-Scholes model with deterministic time-dependent volatility.
- If $\sigma_t > 0$ and for $T_2 < ... < T_m$ the matrix

$$\begin{pmatrix} \int_t^{T_2} v^2(t,s)ds & \dots & \int_t^{T_2} v^m(t,s)ds \\ \vdots & & \vdots \\ \int_t^{T_m} v^2(t,s)ds & \dots & \int_t^{T_m} v^m(t,s)ds \end{pmatrix} \tag{4}$$

  is nonsingular for a.e. t, then Q is the unique equiv. loc. mart. meas.

Outline of proof

"$\Longrightarrow$"

Itô's respresentation theorem implies $\mathbb{E}^P\left[\frac{dQ}{dP}\middle|\mathcal{F}_t\right] = \mathcal{E}\left(\int bdW\right)_t$, and by Girsanov's theorem $\widetilde{W} := W - \int b_t dt$ is a Q-Brownian motion. Apply now Itô's lemma to the price process $C_t^T = c\left(S_t, \int_t^T X(t,s)ds\right)$ and use the Black-Scholes PDE for c. The drifts must vanish, and calculations then yield (1) - (3).

"$\Longleftarrow$"

Set $\frac{dQ^{T^*}}{dP} := \mathcal{E}\left(\int bdW\right)_{T^*}$ on $\mathcal{F}_{T^*}$, then $W^Q := W - \int b_t dt$ is a Q^{T^*}-Brownian motion on $[0, T^*]$ by the Girsanov theorem. Again calculations show that S and C_t^T are Q-local martingales. ■

Example: Call option

For the payoff $h(S) = (S - K)^+$, we can calculate the partials of c explicitly from the Black-Scholes formula. We obtain for the drift restriction for $\alpha(t, T)$

$$
\begin{aligned}
&\alpha(t, T) = -b_t \cdot v(t, T) \\
&- \ \frac{1}{2}\left(\frac{\log^2(S_t/K)}{\left(\int_t^T X(t,s)ds\right)^2} - \frac{1}{\int_t^T X(t,s)ds} - \frac{1}{4}\right) v(t, T) \cdot \int_t^T v(t, s)ds \\
&+ \ \frac{1}{2}\left(\frac{\log^2(S_t/K)}{\left(\int_t^T X(t,s)ds\right)^3} - \frac{1}{2}\frac{1}{\left(\int_t^T X(t,s)ds\right)^2}\right) X(t, T)\left|\int_t^T v(t, s)ds\right|^2 \\
&+ \left(\frac{\log(S_t/K)}{\int_t^T X(t,s)ds} - \frac{1}{2}\right)\sigma_t v^1(t, T) - \frac{\log(S_t/K)}{\left(\int_t^T X(t,s)ds\right)^2} X(t, T)\sigma_t \int_t^T v^1(t, s)ds.
\end{aligned}
$$

Interlude: On strong solutions of SDEs

- **Aim:**
  We want to construct a framework for infinite-dimensional SDEs like the above system

$$\begin{aligned} dS_t &= \mu_t S_t dt + \sigma_t S_t dW_t^1, & S_0 &= s_0 \\ dX(t,T) &= \alpha(t,T,X)dt + v(t,T,X)dW_t, & X(0,T) &= X_0(T) \end{aligned}$$

for all $T \in [0, T^*]$.

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  for all $T \in [0, T^*]$.

- **Idea:**
  Consider 2-dim processes (X, Y) on a space $\widetilde{\Omega} = [0, T^*] \times \Omega$ such that $X(t, T, \omega)$ represents the T-forward impl. vol. and $Y(t, T, \omega)$ is does not depend on T and represents the log-price of stock at time t when the market is in state $\omega \in \Omega$.

Define $\mathcal{S}_c^p$ to be the space of all adapted (w.r.t. $\mathcal{B}[0,T^*]\otimes\mathcal{F}_t$), continuous processes $Z=(Z(t))_{0\le t\le T_0}$ on $\widetilde{\Omega}$ which satisfy

$$\|Z\|^p := \mathbb{E}\left[\frac{1}{T^*}\int_0^{T^*}\sup_{0\le t\le T_0}|Z(t,T)|^p dT\right] < \infty.$$

A function of the form $(t,T,\omega,Z)\to f(t,T,Z(t,\cdot,\omega))$ satisfies *condition (L)* if we have $f=f_1\cdots f_n$ and for all $Z,Z'\in\mathcal{S}_c^p$

- $|f_1(t,T,Z)-f_1(t,T,Z')|\le C|Z(t,T)-Z'(t,T)|$,
- $|f_j(t,T,Z)-f_j(t,T,Z')| \ \le$

$$\le C_j\left(Z(t,t),Z'(t,t),\int_0^{T^*}|Z(t,s)|ds,\int_0^{T^*}|Z'(t,s)|ds\right)$$
$$\times\left(|Z(t,t)-Z'(t,t)|+\int_0^{T^*}|Z(t,s)-Z'(t,s)|ds\right)$$

for $j=2,...,n$, where C_j is a polynomial.

Theorem

Let p be sufficiently large and $(X_0, Y_0) \in L^\infty[0, T^*]$. Suppose that β and ν satisfy condition (L), and for $f \in \{\beta, \nu\}$ we have $|f(u, T, \cdot, (X_0, Y_0))| \leq C$ as well as

$$\mathbb{E}\left[\int_0^{T^*} \int_0^{T_0} \left|f\left(u, T, \cdot, (X, Y)\right)\right|^p du\, dT\right] \leq C(1 + \|(X, Y)\|^p)$$

for $(X, Y) \in \mathcal{S}_c^p$. Then the SDE system

$$d(X, Y)(t, T, \cdot) = \beta(t, T, \cdot, (X, Y))dt + \nu(t, T, \cdot, (X, Y))dW_t$$

with $(X, Y)(0, T, \cdot) = (X_0(T), Y_0(T))$ has a unique solution $(X, Y) \in \mathcal{S}_c^p$.

Idea of proof

1. Define suitable global and local Lipschitz conditions on $\mathcal{S}_c^p$.
2. Suppose that β, μ are globally Lipschitz in the above sence. Then the result is proved by a fixed point argument in $\mathcal{S}_c^p$.
3. Consider locally Lipschitz coefficients without the linear growth condition. By a suitable truncation, locally Lipschitz coefficients become globally Lipschitz. This allows to construct a solution up to an explosion time.
4. Under the linear growth condition the explosion time is infinite.
5. Show that (L) implies local Lipschitz continuity on $\mathcal{S}_c^p$.

A class of arbitrage-free implied volatility models

For the forward implied volatility SDE

$$dX(t,T) = \alpha(t,T)dt + v(t,T)dW_t,$$

we now take coefficients $v(t,T) = (v_1(t,T), ..., v_m(t,T))$ of the form

$$v_j(t,T,X,S) = X(t,T) \cdot V_j\left(t, T, \int_t^T X(t,s)ds, X(t,t), \log S(t,t)\right).$$

Theorem

Let $p \geq 1$ *be sufficiently large. Suppose that* v *is of the above form where* V_j *is Lipschitz and satisfies*

$$V_j(t,T,w,x,y) = V_j(t,t,\epsilon,x,y) \quad \forall w \leq \epsilon, \ \forall T,$$

$$x - \frac{1}{4}(y - \log K)^2 \sum_{j=2}^{m} V_j\,(t,t,0,x,y)^2 \geq x\epsilon \quad \forall t,$$

$$|V_j(t,T,w,x,y)| \leq C \frac{1}{w + (1+\sqrt{x})(1+|y|)} \quad \forall t, T.$$

Then the SDE system for S, X *has a unique solution with* $(\log S, X) \in \mathcal{S}_c^p$. S *does not depend on* T, *and* $X > 0$.

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Thank you very much for your attention!