Inflation-Linked Securities in a Stochastic Monetary Economy

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In the fall of 1972 President Nixon announced that the rate at which inflation was increasing was decreasing.

This was the first time that a sitting US President used a third derivative to advance his case for re-election.

Interest rate models and asset pricing

To begin, let's say what we mean by an interest rate model.

An interest rate model consists of the following ingredients:

(i) A probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with a "market" filtration $\{\mathcal{F}_t\}_{0\leq t<\infty}$ to which all asset price processes are adapted.

(ii) A process $\{B_t\}_{0\leq t<\infty}$ which we call the "nominal money market account" process, and is assumed to be of the form

$$B_t = \exp\left(\int_0^t r_s \mathrm{d}s\right), \tag{1}$$

where $\{r_t\}$ is called the "short rate". We assume $\{r_t\}$ is strictly positive.

(iii) A system of nominal discount bond price processes $\{P_{tT}\}_{0\leq t\leq T<\infty}$. For fixed T we call $\{P_{tT}\}_{0\leq t\leq T}$ the price process of the bond with maturity T. We require that $P_{TT} = 1$ for all T, that $0 < P_{tT} < 1$ for $0 \leq t < T$.

(iv) A pricing kernel $\{\pi_t\}_{0\leq t<\infty}$ (also known as a "state price density", "stochastic discount factor", etc.) with the property that $\{\pi_t B_t\}$ and $\{\pi_t P_{tT}\}$ are martingales with respect to the given measure and filtration.

More generally, for any asset with price process $\{S_t\}_{0\leq t\leq\infty}$, and continuous dividend process $\{D_t\}$ we require that the process $\{M_t\}$ defined by

$$M_t = \pi_t S_t + \int_0^t \pi_s D_s \mathrm{d}s. \tag{2}$$

is a martingale.

We shall assume that the market filtration is generated by a multi-dimensional Brownian motion.

Models for inflation

The modelling of inflation has a number of rather subtle features associated with it.

The essential idea here is that one treats the CPI as if it were the price of a foreign currency.

The associated "foreign" interest rate system then has the interpretation of being the system of "real" interest rates.

The basic setup can be described as follows.

Let us write $\{\pi_t^N\}$ now for the nominal pricing kernel, and $\{P_{tT}^N\}$ for the nominal discount bond system.

For the nominal discount bond system we have

$$P_{tT}^N = \frac{\mathbb{E}_t[\pi_T^N]}{\pi_t^N}. \tag{3}$$

The associated dynamics are given by:

$$\frac{\mathrm{d}P_{tT}^N}{P_{tT}^N} = \left(r_t^N + \lambda_t^N \Omega_{tT}^N\right) \mathrm{d}t + \Omega_{tT}^N \, \mathrm{d}W_t. \tag{4}$$

Here r_t^N (nominal short rate) and λ_t^N (nominal market price of risk) are defined by

$$\frac{\mathrm{d}\pi_t^N}{\pi_t^N} = -r_t^N \mathrm{d}t - \lambda_t^N \mathrm{d}W_t, \tag{5}$$

and Ω_{tT}^N is the nominal discount-bond vector volatility.

The dynamics of the discount bond system gives rise to the interpretation of r_t^N and λ_t^N.

Next we introduce a price index $\{C_t\}$ which we call consumer price index.

The rate of inflation over a given period is defined to be the percentage increase in the price index over the given period, expressed on an annualised basis.

The interesting question is: how do we model $\{C_t\}$?

One is tempted to write down the usual arbitrage-free dynamics of a risky asset, but since $\{C_t\}$ is a price index (including the prices of non-durable items such as the price of a meal in a restaurant) one cannot immediately conclude that the usual arbitrage-free dynamics apply.

Instead, we consider a related family of asset prices: the prices of index-linked bonds.

Index-linked bonds

An index-linked discount bond pays out the amount C_T at time T.

Thus we can think of such a bond as a kind of derivative.

The value P^I_{tT} of an index-linked bond at time t is therefore

$$P^I_{tT} = \frac{\mathbb{E}_t[\pi^N_T C_T]}{\pi^N_t}. \tag{6}$$

If we divide P^I_{tT} by C_t we get the value (in *real* terms) of a bond that pays one unit of goods and services at time T:

$$P^R_{tT} = \frac{\mathbb{E}_t[\pi^N_T C_T]}{\pi^N_t C_t}. \tag{7}$$

Now suppose we define the real pricing kernel π^R_t by writing

$$\pi^R_t = \pi^N_t C_t. \tag{8}$$

Then clearly for the price index we have

$$C_t = \frac{\pi^R_t}{\pi^N_t}. \tag{9}$$

In terms of π_t^R, the real discount bond prices take the form

$$P_{tT}^R = \frac{\mathbb{E}_t[\pi_T^R]}{\pi_t^R}. \tag{10}$$

Now suppose we define r_t^R and λ_t^R by the relation

$$\frac{\mathrm{d}\pi_t^R}{\pi_t^R} = -r_t^R \mathrm{d}t - \lambda_t^R \mathrm{d}W_t. \tag{11}$$

Then it follows that the dynamics of the real discount bond system are of the form:

$$\frac{\mathrm{d}P_{tT}^R}{P_{tT}^R} = \left(r_t^R + \lambda_t^R \Omega_{tT}^R\right) \mathrm{d}t + \Omega_{tT}^R \, \mathrm{d}W_t. \tag{12}$$

Here Ω_{tT}^R is the real discount bond volatility.

This shows r_t^R and λ_t^R have the interpretation of being the real short rate, and the real market price of risk, respectively.

The CPI process

Now we can deduce the dynamics of the price index C_t.

In particular, by an application of the Ito calculus we obtain:

$$\frac{\mathrm{d}C_t}{C_t} = \left[r_t^N - r_t^R + \lambda_t^N \left(\lambda_t^N - \lambda_t^R\right)\right] \mathrm{d}t + \left(\lambda_t^N - \lambda_t^R\right) \mathrm{d}W_t. \tag{13}$$

We note that the consumer price index volatility is the difference between the nominal and real risk premium vectors. Thus we can write

$$\frac{\mathrm{d}C_t}{C_t} = \left[r_t^N - r_t^R + \lambda_t^N \nu_t\right] \mathrm{d}t + \nu_t \mathrm{d}W_t. \tag{14}$$

Here the CPI volatility vector is defined by:

$$\nu_t = \lambda_t^N - \lambda_t^R. \tag{15}$$

The instantaneous rate of inflation I_t is by definition the drift of the consumer price index. Thus we have:

$$I_t = r_t^N - r_t^R + \lambda_t^N \nu_t. \tag{16}$$

With these relations at hand, we are now in a position to build models for pricing inflation-linked derivatives.

Monetary systems

Inflation is a monetary phenomenon.

To model a monetary economy with inflation we need three more ingredients.

These are: the rate of real consumption $\{k_t\}$, the money supply $\{M_t\}$, and the liquidity benefit rate $\{\beta_t\}$ associated with the money supply.

The process $\{\beta_t\}$ describes the effective rate of benefit, in dollars per unit of time, associated with a given level of the money supply, on an aggregate basis.

If the money supply at time t is M_t, the associated benefit rate is $\beta_t M_t$ dollars per year.

Thus one thinks of the benefit as a kind of "consumable" commodity.

Of course what really "counts" is the *real* rate of benefit derivable from the presence of the money supply.

This is given by $l_t = \beta_t M_t / C_t$.

The consumer price index is thus used as a kind of "exchange rate" to convert the nominal benefit into a real benefit.

Our assumption is that in equilibrium the price level $\{C_t\}$ and the rate of real consumption $\{k_t\}$ will be adjusted to each other in such a way as to maximise the total expected utility.

Optimisation

We assume the existence of a standard bivariate utility function $U(x,y)$ such that the expected total utility

$$\mathbb{E}\left[\int_0^T \mathrm{e}^{-\gamma t} U(k_t, l_t)\mathrm{d}t\right] \quad (17)$$

is maximised over some time horizon.

This is subject to the budget constraint

$$W_0 = \mathbb{E}\left[\int_0^T \pi_t^N C_t k_t \mathrm{d}t + \int_0^T \pi_t^N \beta_t M_t \mathrm{d}t\right]. \quad (18)$$

Here W_0 denotes the total wealth of the economy (including, e.g., the present value of any forthcoming income) available for the given period.

A standard argument shows that the maximum is achieved if

$$U_x(k_t, l_t) = \mu \mathrm{e}^{\gamma t} \pi_t^N C_t, \qquad U_y(k_t, l_t) = \mu \mathrm{e}^{\gamma t} \pi_t^N C_t \quad (19)$$

for some constant μ which is fixed by the budget constraint.

These equations can then be solved to give relations between k_t, l_t, C_t, and π_t^N.

In particular, if we specify any two of k_t, l_t, C_t, and π_t^N, then the other two can be determined.

The idea is that for modelling purposes we regard the consumption (or GDP) and the money supply processes as being specified exogenously.

Then C_t and π_t^N are determined, and we can work out π_t^R.

A simple example

A simple choice for $U(x,y)$ is given by

$$U(x,y) = A\ln(x) + B\ln(y), \tag{20}$$

where A, B are constants.

It follows that

$$U_x(k_t,l_t) = \frac{A}{k_t}, \qquad U_y(k_t,l_t) = \frac{B}{l_t}. \tag{21}$$

Equating these we get

$$k_t = \frac{A}{B}\, l_t = \frac{A}{B}\,\frac{\beta_t M_t}{C_t}. \tag{22}$$

Hence, rearranging, we have

$$C_t = \frac{A}{B}\,\frac{\beta_t M_t}{k_t}. \tag{23}$$

Thus the consumer price index is determined completely by the nominal money supply benefit and the rate of real consumption.

But we can also determine the pricing kernels. A calculation shows that

$$\pi_t^N = \frac{B\mathrm{e}^{-\gamma t}}{\mu\beta_t M_t}, \qquad \pi_t^R = \frac{A\mathrm{e}^{-\gamma t}}{\mu k_t}. \tag{24}$$

Thus both the real and the nominal interest rate systems are determined by the model.

Now suppose that H_T is the payout of a contingent claim (e.g. an inflation derivative).

Then in this model we have

$$H_0 = \beta_0\, M_0\, \mathrm{e}^{-\gamma t}\, \mathbb{E}\left[\frac{H_T}{\beta_T M_T}\right]. \tag{25}$$

This example indicates how expectations concerning monetary policy can have a direct effect on the valuation of derivatives.

We see therefore how with a little mathematical reasoning we can model the relations between interest rates, inflation, consumption (or GDP), and the money supply, and draw some interesting conclusions.

References

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