Eidgenössische Technische Hochschule Zürich
Swiss Federal Institute of Technology Zurich

Dynamic Credit Portfolio Derivatives Pricing

Philippe Ehlers

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Agenda

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A. The Building Blocks of STCDOs

Obligors and Loss Process

- $j = 1, \dots, N$ obligors
- default indicator processes $D_j(t) := \mathbf{1}_{\{\text{obligor } j \text{ has defaulted until } t\}}$
- All losses given default are normalized to one (zero recovery).
- No joint defaults

The key quantity of the model is the **portfolio loss process**

$$L_t := \sum_{j=1}^{N} D_j(t)$$

Definition. The n^{th} loss times $\tau_n := \inf\{t > 0;\ L_t \geq n\}$

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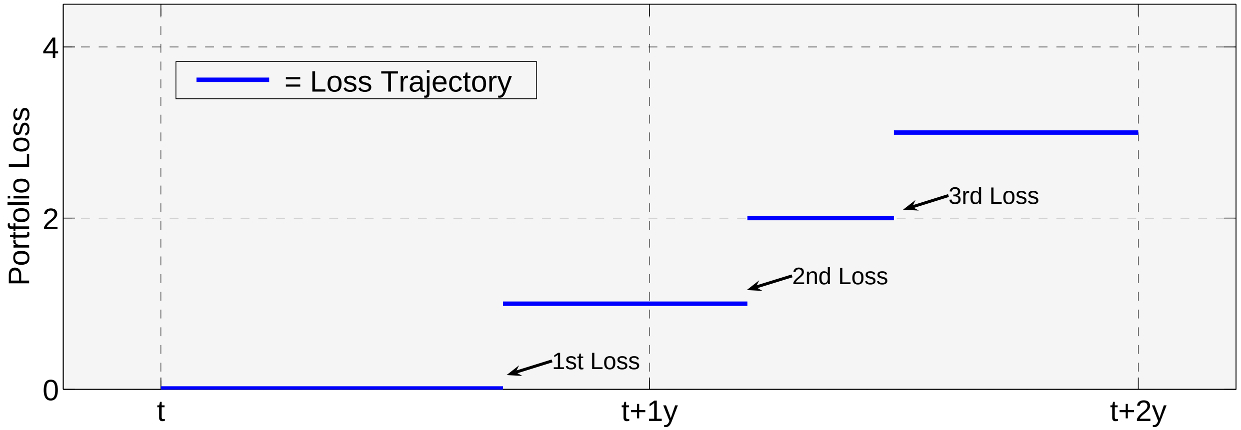
STCDOs: An Example

- Portfolio of 4 credits, each with 1$ notional



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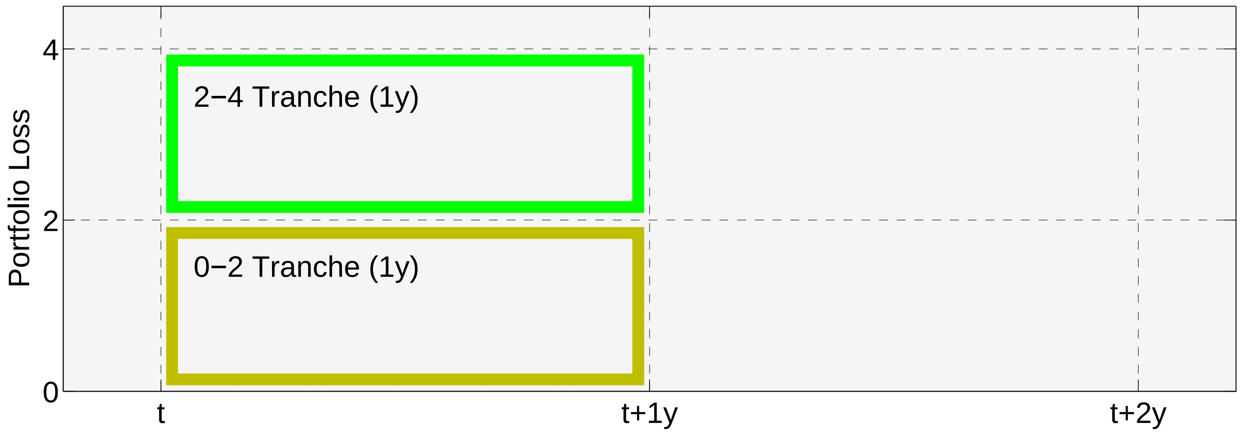

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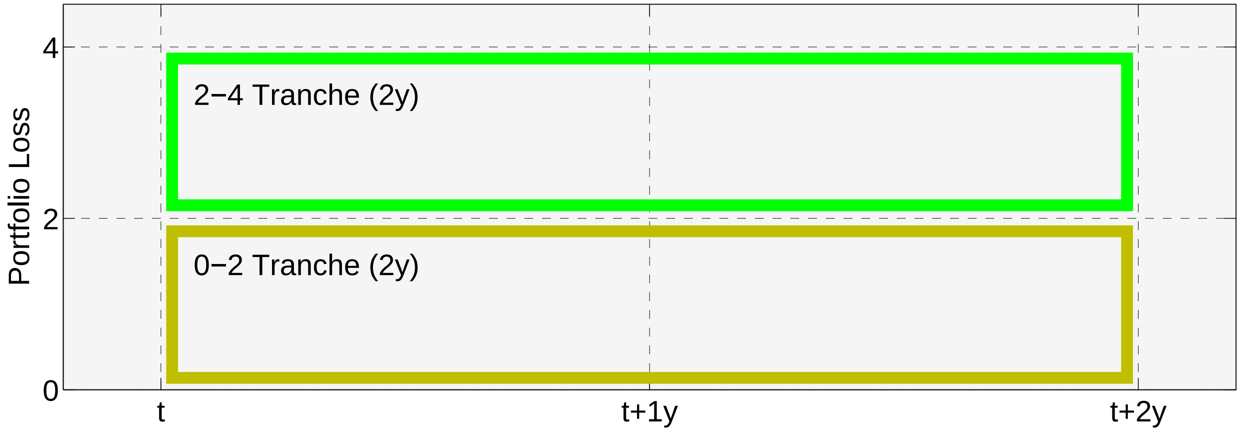

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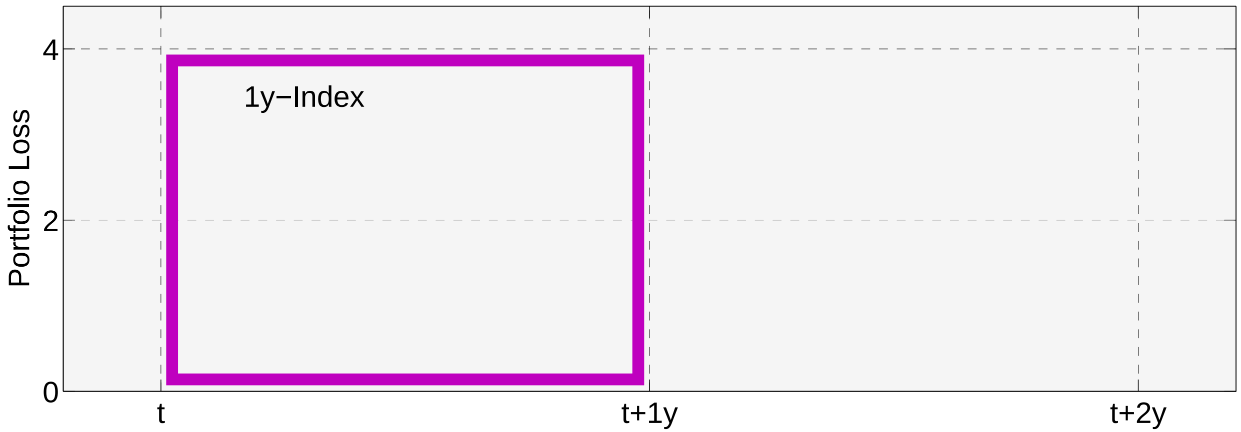

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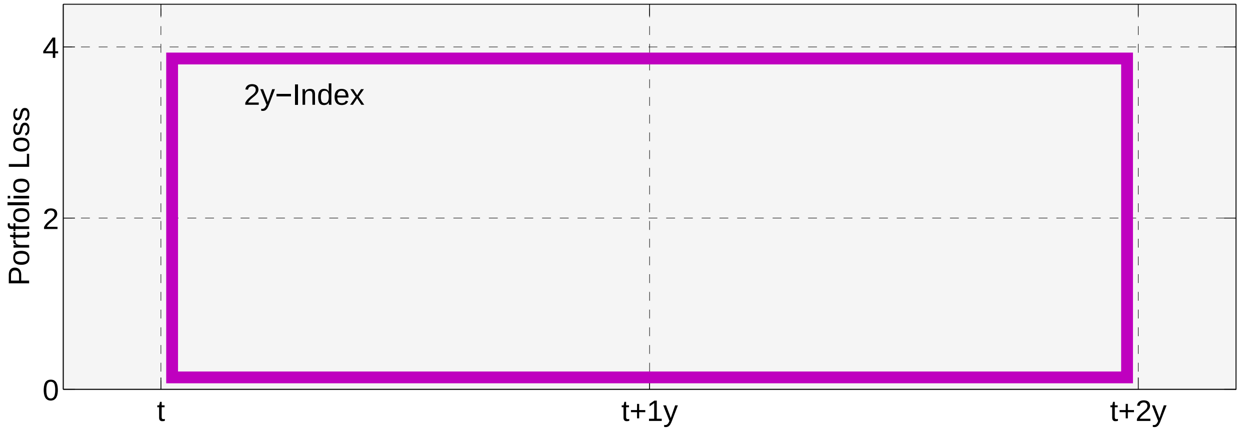

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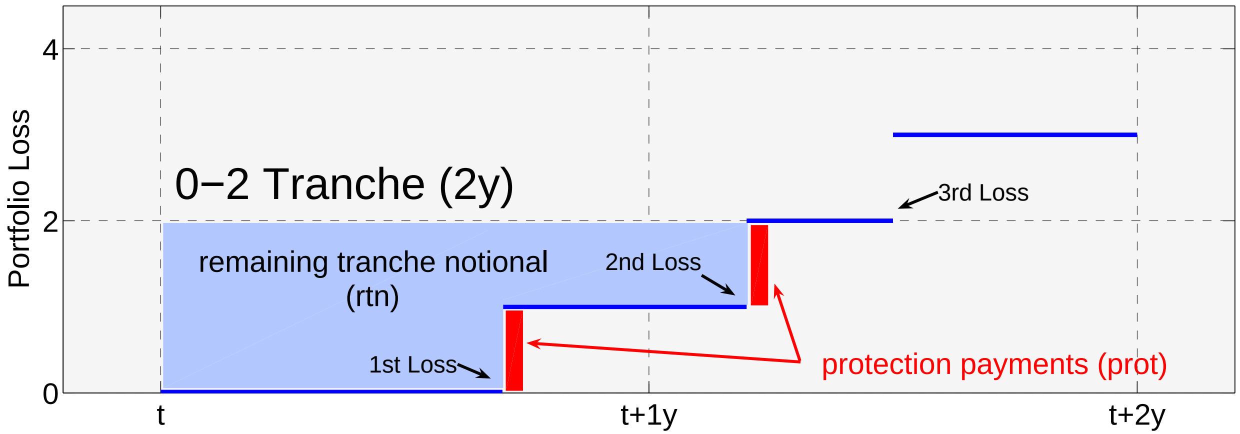

- STCDO swaps "tranche losses" into "tranche spread × remaining tranche notional"

- Tranche/Index spread $\overline{s}$ is chosen such that

$$\overline{s} \times NPV(\text{rtn}) = NPV(\text{prot}) \qquad \Longrightarrow \qquad \overline{s} = \frac{NPV(\text{prot})}{NPV(\text{rtn})}$$

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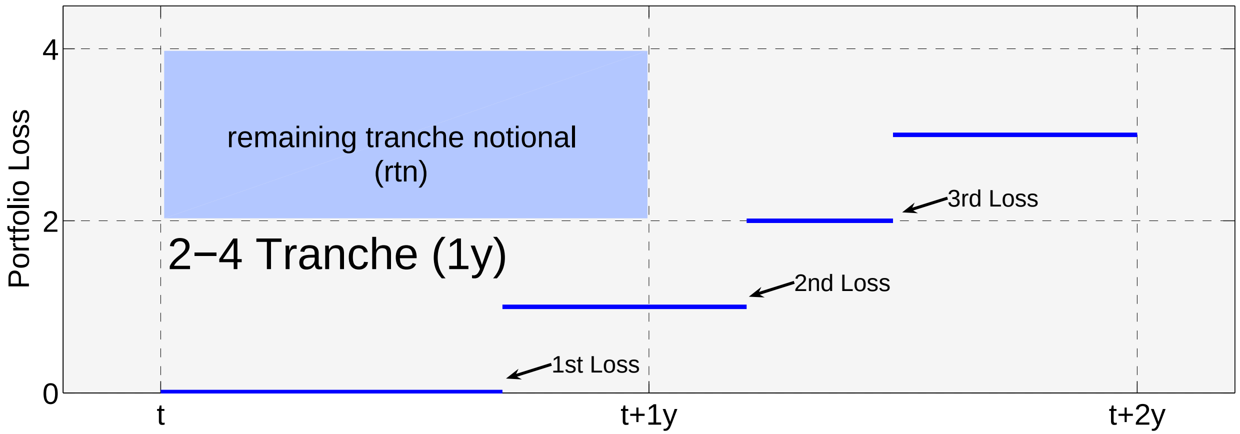

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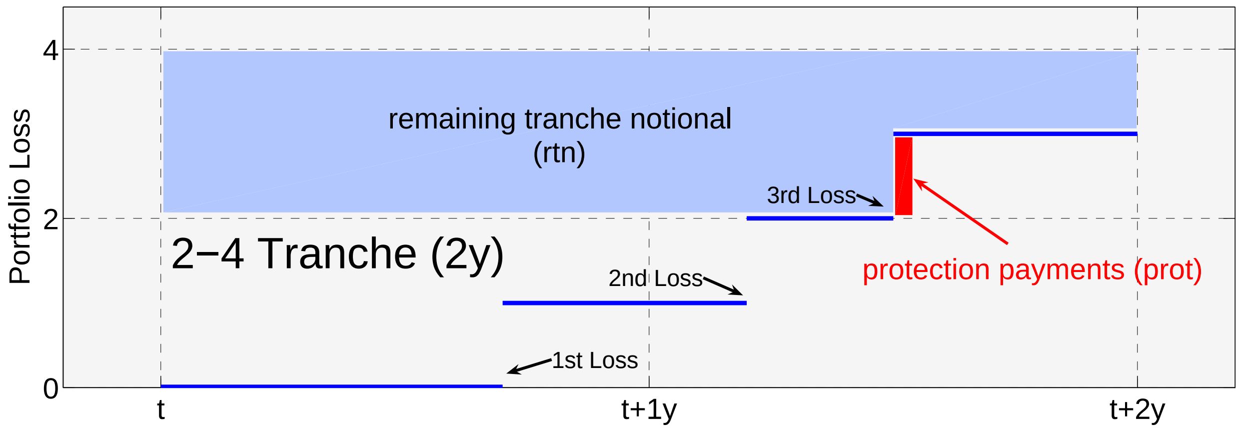

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Market Quotes: STCDOs on iTraxx Europe

Maturity		3Y		5Y		7Y		10Y	
Low	High	Bid	Ask	Bid	Ask	Bid	Ask	Bid	Ask
0	3	6.0	7.5	29.50	30.25	47.1	48	58.25	59.25
3	6	18	28	96	100	193	200	505	520
6	9	6	13	33	36	52	57	100	106
9	12			13	15	29	34	48	55
12	22			7.50	8.75	12	15	22	25
22	100			2.25	4.00	5.25	7.25	8.25	10.75
Index		22		38		47		58	

Quotes for loss protection on tranches of European iTraxx Series 4, on Sept. 26th, 2005. Lower and upper attachment points are in % of notional, base correlation (BC) is given in %. Prices for the 0-3 tranche are % of notional upfront plus 500bp running, all other prices are bp p.a.. Source: JPMorgan.

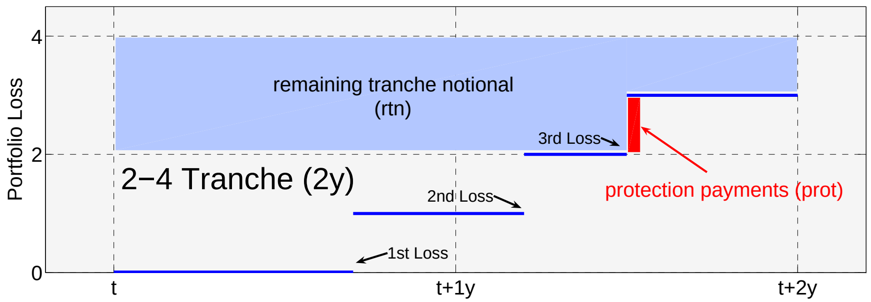
2N+1 Building Blocks

Assumption. These instruments are traded at all times t for all $T \geq t$

$$
\begin{array}{lllll}
P_n(t,T) & \text{pays } 1\$ & \text{at } T & \text{iff } L_T = n, & \text{for } n = 0,\dots,N \\
U_n(t,T) & \text{pays } 1\$ & \text{at } \tau_n & \text{iff } \tau_n \in (t,T], & \text{for } n = 1,\dots,N
\end{array}
$$

STCDO Pricing. For any tranche and at all times t

- NPV(prot) can be expressed using the $U_n(t,T)$s
- NPV(rtn) can be expressed using the $P_n(t,T)$s



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B. Initial No-Arbitrage Conditions

(⋆P) $P_n(t,T) \geq 0$ and $P_n(t,t) = \mathbf{1}_{\{L_t=n\}}$

(⋆U) $U_n(t,T) \geq 0$ and nondecreasing in T and $U_n(t,t) = 0$

The default-free ZCB $B(t,T) = \sum_{n=0}^{N} P_n(t,T)$ requires

(⋆B) $B(t,T) \in (0,1]$ and non-increasing in T and $B(t,t) = 1$

One more condition: Let $\Pi_n(t,T) := P_n(t,t) + U_n(t,T) - U_{n+1}(t,T) - P_n(t,T)$

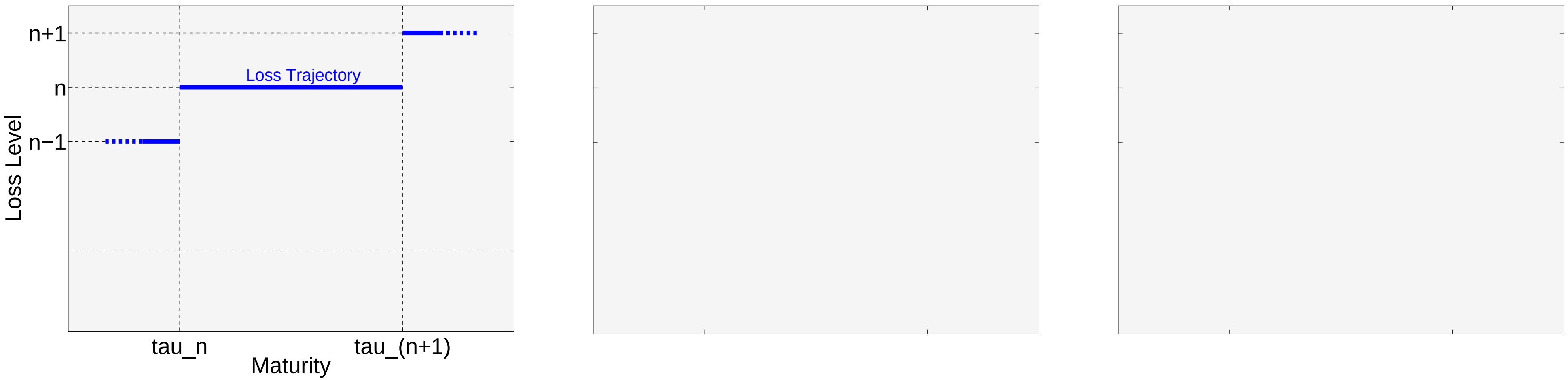
(⋆Π) $\Pi_n(t,T)$ nondecreasing in T

Why $\Pi_n(t,T)$ nondecreasing in T?

Let $\delta > 0$ and recall $\Pi_n(t,T) := P_n(t,t) + U_n(t,T) - U_{n+1}(t,T) - P_n(t,T)$

A. $L_u \neq n$ for all $u \in [T, T+\delta)$ $\Longrightarrow$ $\Pi_n(t,T+\delta) - \Pi_n(t,T)$ pays *nothing*

B. $L_u = n$ for some $u \in [T, T+\delta)$ $\Longrightarrow$ $\Pi_n(t,T+\delta) - \Pi_n(t,T)$ pays...



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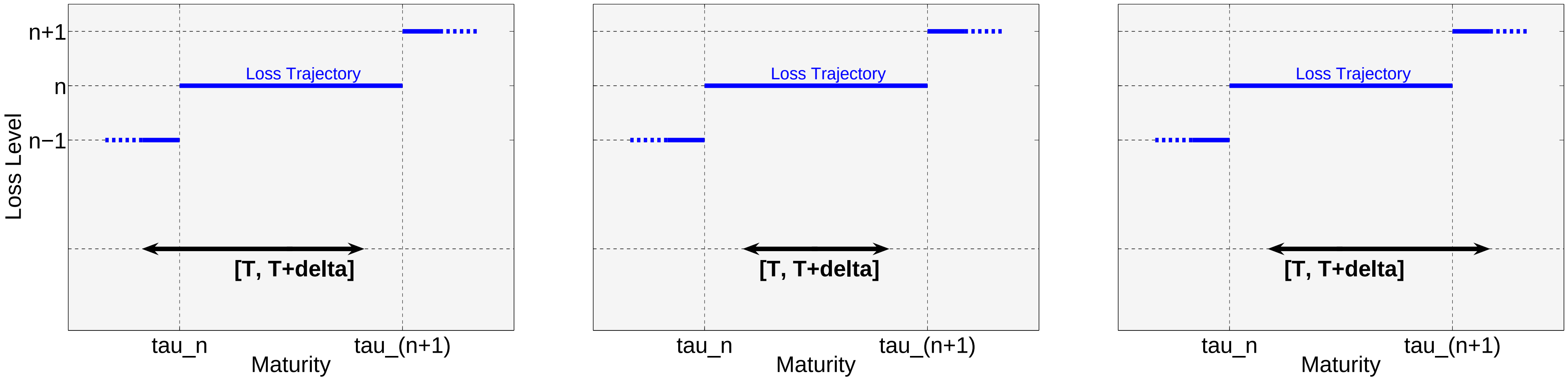

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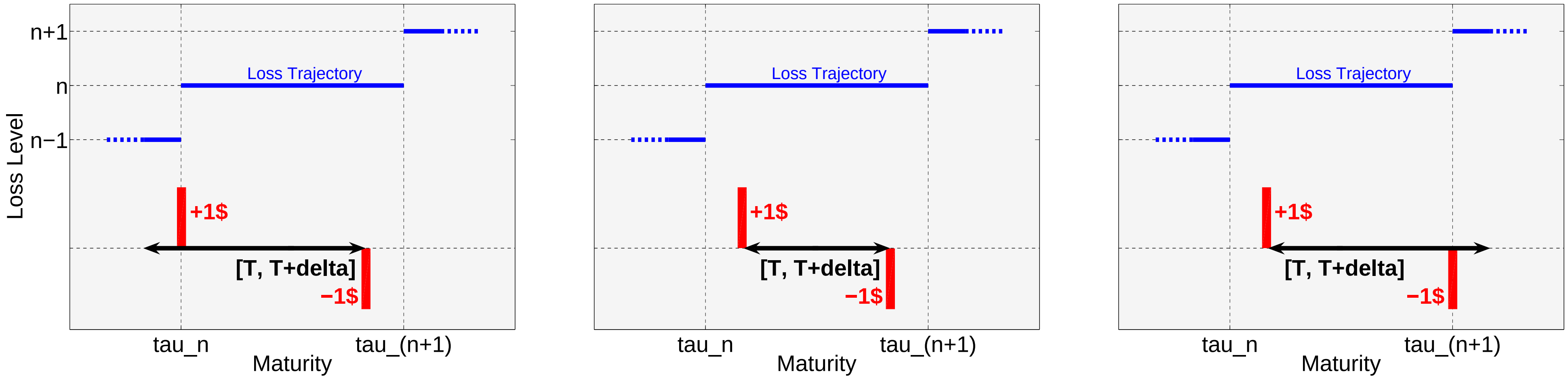

$\Longrightarrow$ $(\star\Pi)$ $\Pi_n(t,T)$ and nondecreasing in T

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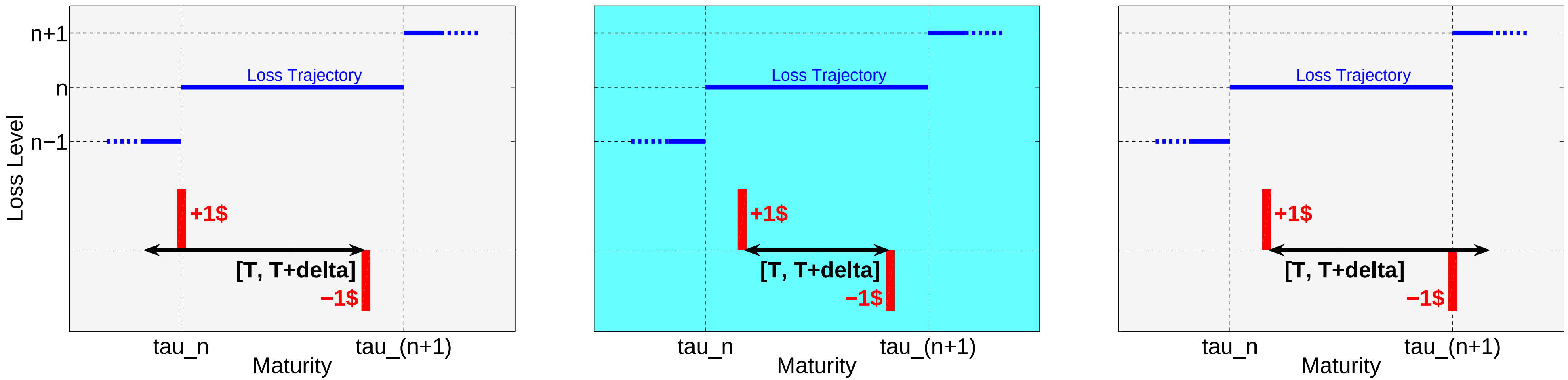

$\Longrightarrow$ $(\star\Pi)$ $\Pi_n(t,T)$ and nondecreasing in T

Loss-Contingent Forward Rates

For $\delta > 0$ small

- $\Pi_n(t, T+\delta) - \Pi_n(t,T)$ pays interest over $[T, T+\delta)$ *if and only if* $L_T = n$
- $U_n(t, T+\delta) - U_n(t,T)$ protects against loss in $[T, T+\delta)$ *if and only if* $L_T = n$

Lemma. Let $P_n(t,T)$, $U_n(t,T)$ be in $\mathcal{C}^1(T)$ (+ tech. conditions). Then the ($\star\cdot$)s hold *if and only if* there exist $f_n(t,T)$, $F_n(t,T) \geq 0$ such that

$$\partial_T P_n = -P_n(F_n + f_n) + P_{n-1}F_{n-1} \qquad P_n(t,t) = \mathbf{1}_{\{L_t = n\}}$$

$$\partial_T U_n = \quad P_{n-1}F_{n-1} \qquad U_n(t,t) = 0$$

- $f_n(t,T)$ is *loss-contingent* T-forward interest rate, conditional on $L_T = n$
- $F_n(t,T)$ is *loss-contingent* T-forward loss rate, conditional on $L_T = n$
- Under NAB, instead of P_n, U_n, simply model f_n, $F_n \geq 0$ and L_t!

Forward Interest Rates

Let $f(t,T)$ be the forward interest rate (i.e. $B(t,T) = e^{-\int_t^T f(t,u)du}$)

Special Case. If $f_n(t,T) = f(t,T)$ for all n, then $P_n(t,T) = B(t,T)p_n(t,T)$, where

$$\partial_T p_n = -p_n F_n + p_{n-1} F_{n-1} \qquad p_n(t,t) = \mathbf{1}_{\{L_t = n\}}$$

and there is a one-to-one mapping between (cf. Schönbucher (05))

$$\underbrace{f \text{ and } N \text{ STCDO spreads}}_{\text{N+1 forward curves}} \quad \text{and} \quad \underbrace{f \text{ and } F_0, \ldots, F_{N-1}}_{\text{N+1 forward curves}}$$

Remark. Large losses in credit index portfolio indicate macroeconomic crises and should hence influence interest rates (via investor preferences and/or monetary policies).

C. Dynamic No-Arbitrage

General Setup. Probability space $(\Omega, \mathcal{F}, \mathbf{P})$, horizon $T^* < \infty$, filtration $(\mathcal{F}_t)_{t\in[0,T^*]}$ satisfies the usual conditions. All processes are $(\mathcal{F}_t)$-adapted and càdlàg.

Assumption.

- The $(\star\cdot)$s hold for all $t \in [0, T^*]$ and we may directly model f_n, $F_n \geq 0$ and L_t.
- L_t admits an intensity $\lambda_t^{\mathbf{P}}$ under $\mathbf{P}$, i.e. $\overline{L}_t := L_t - \int_0^t \lambda_s^{\mathbf{P}} ds$ is a $\mathbf{P}$-martingale.
- $b_t = e^{\int_0^t r_s ds}$ is the money market account and $\beta_t := 1/b_t$.

Absence of dynamic arbitrage opportunities $\iff$ There exists an EMM $\mathbf{Q} \sim \mathbf{P}$.

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EMM Relationship

Theorem. Let $\lim_{T\downarrow T_0} \mathbf{E}[\,|x_n(t,T) - x_n(t,T_0)|\,] = 0$ for all t, T_0, n and $x = f, F$.

Then $\mathbf{P}$ is an EMM *if and only if*

(i) $\beta_t P_n f_n(t,T)$ and $\beta_t P_n F_n(t,T)$ are $\mathbf{P}$-martingales for all n and T,

(ii) $r_t = f_{L_t}(t,t)$ and $\lambda_t^{\mathbf{P}} = F_{L_t}(t,t)$.

Corollary. Under the conditions above, let $\mathbf{P}$ be an EMM. Then

$$f_n(t,T) = \mathbf{E}^{\mathbf{P}_n^T}[\,r_T \mid \mathcal{F}_t\,] \qquad \text{and} \qquad F_n(t,T) = \mathbf{E}^{\mathbf{P}_n^T}\left[\,\lambda_T^{\mathbf{P}} \mid \mathcal{F}_t\,\right]$$

where $\mathrm{d}\mathbf{P}_n^T = \frac{\beta_T \mathbf{1}_{\{L_T = n\}}}{P_n(0,T)} \mathrm{d}\mathbf{P}$.

D. An Example: Loss-HJM Model

Model Setup. We assume

$$\begin{aligned} df_n(t,T) &= \alpha_n^f(t,T)dt + \sigma_n^f(t,T)dW \\ dF_n(t,T) &= \alpha_n^F(t,T)dt + \sigma_n^F(t,T)dW \end{aligned}$$

Lemma. (tech. conditions) $dP_n(t,T) = u_n(t,T)dt + v_n(t,T)dW + \phi_n(t,T)d\overline{L}$

Theorem [EMM]. $\mathbf{P}$ is an EMM *if and only if* $\lambda_t^{\mathbf{P}} = F_{L_t}(t,t)$, $r_t = f_{L_t}(t,t)$ and

$$\alpha_n^f = \frac{v_n' \sigma_n^f}{P_n} \qquad \text{and} \qquad \alpha_n^F = \frac{v_n' \sigma_n^F}{P_n} \qquad \text{on } \{P_n > 0\}$$

Remark. Even though here $f_n(t,T)$ and $F_n(t,T)$ are continuous processes, $\lambda_t^{\mathbf{P}}$ and r_t may jump at loss times τ_n $\implies$

- Contagion is a natural feature of our model.
- Interest rates respond to losses.

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Application

Pricing a Tranche Option with

- Underlying: $[a,b]$-tranche spread with maturity T_2.
- Strike: K
- Maturity: $T_1 < T_2$,

The value of the option at T_1 is

$$\left(\bar{s}_a^b(T_1,T_2) - K\right)_+ \times \mathrm{rtn}_a^b(T_1,T_2) \quad \overset{!}{=} \quad g\big(L_{T_1}, f_n(T_1,T_2), F_n(T_1,T_2),\, n \geq 0\big)$$

The option value today ($t=0$) is

$$\mathbf{E}\left[\, g\big(L_{T_1}, f_n(T_1,T_2), F_n(T_1,T_2),\, n \geq 0\big) \,\right]$$

Advantage. Using Monte-Carlo, simulation is only required until T_1 (not until T_2)! But (additional to f_n and F_n) one has to simulate L_t.

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Outlook

Canonical Loss Model. Using a "default-free" market filtration $(\mathcal{G}_t)_{t\geq 0}$, we construct

a *canonical* loss process L_t.

- The canonical loss process L_t can be interpreted as a generalized Cox process or $\mathcal{G}_\infty$-conditional Markov chain
- If "adding" a general loss process L_t to a "default-free" market filtration satisfies $\mathbb{H}$ hypothesis, then L_t is a canonical loss process.
- In the canonical model, we *never* need to simulate the loss process for derivatives pricing $\Longrightarrow$ Extremely fast and accurate simulation.

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E. Conclusion

- Our credit portfolio model automatically calibrates to the full STCDO term structure.
- We identify natural building blocks $P_n(t,T)$ and $U_n(t,T)$ to express/price all STCDOs.
- $P_n(t,T)$ and $U_n(t,T)$ are parameterized using the spot loss proces L_t,
  - ⋆ *loss-contingent* forward interest rates $f_n(t,T)$ and
  - ⋆ *loss-contingent* forward loss rates $F_n(t,T)$.
  - ⋆ Initial NAB conditions: $f_n,\ F_n \geq 0$.
- $f_n(t,T)$ and $F_n(t,T)$ have a clear financial interpretation.
- We study the model under an EMM.
- For applications, we propose a Loss-HJM specification of the model.